

# in potential approach of NRQCD

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## Abstract

We consider a general scheme for calculating the leptonic constant of heavy quarkonium  $\bar{Q}Q$  in the framework of nonrelativistic quantum chromodynamics, NRQCD, operating as the effective theory of nonrelativistic heavy quarks. We explore the approach of static potential in QCD, which takes into account both the evolution of effective charge in the three-loop approximation and the linearly raising potential term, which provides the quark confinement. The leptonic constants of  $b\bar{b}$  and  $c\bar{c}$  systems are evaluated by making use of two-loop anomalous dimension for the current of nonrelativistic quarks, where the factor for the normalization of matrix element is introduced in order to preserve the renormalization group invariance of estimates.

## 1 Introduction

A description of leptonic constants for a heavy quarkonium composed of heavy quark and heavy anti-quark  $\bar{Q}Q$  plays an important role in studies of QCD dynamics in the sector of heavy quarks. So, the precision calculations of heavy quark masses  $m_{b,c}$  and running coupling constant  $\alpha_s$  in QCD are performed in the framework of QCD sum rules [1] and effective theory of nonrelativistic heavy quarks, NRQCD [2]. In such the approach the experimental data on both the masses of  $S$ -wave quarkonium states and their leptonic constants [3] are used. The consideration of NRQCD sum rules was done for the bottomonium with taking into account of one-loop gluon corrections in [4] (next-to-leading-order, NLO) as well as with two-loop corrections in [5–7] (next-to-next-to-leading-order, N<sup>2</sup>LO), and for the charmonium in [8]. In general, the sum rules of NRQCD could be used for the solution of inverse problem, i.e. the calculation of leptonic constants at given values of heavy quark masses, a definite normalization of coupling constant and the experimental mass spectrum of quarkonia. The only additional entry is caused by that the analysis in the framework of NRQCD sum rules supposes that the region under study remains correctly defined only if one takes into account the higher excitations of ground state, so that one should fix the ratios of leptonic constants, i.e. their relative weights in the sum rules, along with the mass spectrum. Another way is to go in the region of parameters, where the contributions of ground

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state excitations are negligible, and the term of gluon condensate becomes significant, while its Wilson coefficient requires cumbersome calculations in higher orders of perturbation theory.

An alternative method is the description of heavy quarkonium in the framework of potential approach. In this way, if the static potential in QCD with infinitely heavy sources in the fundamental representation of color gauge group is given, then in the leading order of NRQCD, i.e., if we neglect relativistic corrections and interactions depending on the quark spins, the heavy quark masses can be determined with a high accuracy from the comparison of experimental mass spectrum of heavy quarkonia  $\bar{b}b$  and  $\bar{c}c$  with the estimates in the NRQCD. In this procedure, one calculates the spectra of stationary energy levels  $E$  for the nonrelativistic Schrödinger equation with the static potential, so that the masses are defined by the expression  $M = m_1 + m_2 + E$ , where  $m_{1,2}$  are the masses of heavy quark and heavy anti-quark composing the quarkonium. In addition, in ref. [9] authors derived the static potential in QCD, so that it takes into account recent perturbative results at short distances [10,11] as well as the regime of confinement in the infrared region. This potential is straightforward modification of model by Buchmüller and Tye [12], who suggested the method with unified two-loop  $\beta$ -function for the effective charge  $\mathbf{a}$ . This function satisfied two definite asymptotic limits at  $\mathbf{a} \rightarrow 0$  (the perturbation theory) and  $\mathbf{a} \rightarrow \infty$  (the potential confining the quarks in hadrons, in the form of linearly raising term under the distance increase). The calculations in three loops [10,11] shown that the value and sign of coefficient  $\beta_2$  modelled in the paper by Buchmüller and Tye are significantly different from the correct results for the  $\beta$ -function. The modification allows us to reach a consistent description for combined data on the mass spectra of heavy quarkonia and the evolution parameter  $\Lambda_{QCD}$  for the running coupling constant measured at high virtuality of  $m_Z$  [3]. Thus, in the potential approach we have got a possibility to estimate the leptonic constants of heavy quarkonia in NRQCD with the quark masses and coupling constant  $\alpha_s$  adjusted with a high accuracy.

In this way, we consider a general scheme for the calculation of leptonic constants, since the results of NRQCD have to be related with the quantities defined in full QCD. There are two challenges in this respect. The first one is the Wilson coefficient depending on the scale of calculations in NRQCD, which relates the current of nonrelativistic quarks considered in the leading order of inverse heavy quark mass, with the electromagnetic current of heavy quarks in full QCD. An anomalous dimension of this coefficient  $\mathcal{K}$  is known with the two-loop accuracy [13–15]. The factor  $\mathcal{K}$  is matched with the value, which is given by the equality of currents in QCD and its nonrelativistic approximation at a scale  $\mu_{\text{hard}}$  about the mass of heavy quark. Therefore, the operator equality of currents in full QCD and NRQCD is completely defined up to the two-loop accuracy. The second challenge is the matrix element for the current of nonrelativistic quarks over the vacuum and physical state of quarkonium. Because of renormalization group invariance of physical quantities, this matrix element has got the anomalous dimension equal to that of the current in NRQCD. For the first time, in the potential approach we perform the consideration involving the factor  $\mathcal{A}$  providing the renormalization invariance of results. In this way, we have to introduce a scale of normalization for  $\mathcal{A}$ , so that the reasonable region of scale change is restricted by the physical characteristics for the quark system under study. Then, the obtained estimates of leptonic constants for the quarkonia are stable in a region of change for both the QCD-to-NRQCD matching scale  $\mu_{\text{hard}}$  and the point of perturbative calculations in NRQCD  $\mu_{\text{fact}}$ .

In Section 2 we consider the description of leptonic constants for the heavy quarkonia in the framework of NRQCD. The potential approach is actually given in Section 3. The numerical estimates are presented in Section 4. In Conclusion we summarize the obtained results.

## 2 Leptonic constants

In the NRQCD approximation for the heavy quarks, the calculation of leptonic constant for the heavy quarkonium with the two-loop accuracy requires the matching of NRQCD currents with the currents in full QCD,

$$J_\nu^{QCD} = \bar{Q}\gamma_\nu Q, \quad \mathcal{J}_\nu^{NRQCD} = \chi^\dagger \sigma_\nu^\perp \phi,$$

where we have introduced the following notations:  $Q$  is the relativistic quark field,  $\chi$  and  $\phi$  are the nonrelativistic spinors of anti-quark and quark,  $\sigma_\nu^\perp = \sigma_\nu - v_\nu(\sigma \cdot v)$ , where  $v$  is the four-velocity of heavy quarkonium, so that

$$J_\nu^{QCD} = \mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}}) \cdot \mathcal{J}_\nu^{NRQCD}(\mu_{\text{fact}}), \quad (1)$$

where the scale  $\mu_{\text{hard}}$  gives the normalization point for the matching of NRQCD with full QCD, while  $\mu_{\text{fact}}$  denotes the normalization point for the calculations in the perturbation theory of NRQCD.

For the heavy quarkonium composed of heavy quarks with the same flavor, the Wilson coefficient  $\mathcal{K}$  is known with the two-loop accuracy [13–16]

$$\mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}}) = 1 - \frac{8}{3} \frac{\alpha_s^{\overline{\text{MS}}}(\mu_{\text{hard}})}{\pi} + \left( \frac{\alpha_s^{\overline{\text{MS}}}(\mu_{\text{hard}})}{\pi} \right)^2 c_2(\mu_{\text{hard}}; \mu_{\text{fact}}), \quad (2)$$

and  $c_2$  is explicitly given in [14, 15]. The anomalous dimension of factor  $\mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}})$  in NRQCD is defined by

$$\frac{d \ln \mathcal{K}(\mu_{\text{hard}}; \mu)}{d \ln \mu} = \sum_{k=1}^{\infty} \gamma_{[k]} \left( \frac{\alpha_s^{\overline{\text{MS}}}(\mu)}{4\pi} \right)^k, \quad (3)$$

whereas the two-loop calculations<sup>1</sup> give

$$\gamma_{[1]} = 0, \quad (4)$$

$$\gamma_{[2]} = -16\pi^2 C_F \left( \frac{1}{3} C_F + \frac{1}{2} C_A \right). \quad (5)$$

The initial condition for the evolution of factor  $\mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}})$  is given by the matching of NRQCD current with full QCD at  $\mu = \mu_{\text{hard}}$  [14, 15].

The leptonic constant of heavy quarkonium is defined in the following way:

$$\langle 0 | J_\nu^{QCD} | \bar{Q}Q, \lambda \rangle = \epsilon_\nu^\lambda f_{\bar{Q}Q} M_{\bar{Q}Q}, \quad (6)$$

where  $\lambda$  denotes the vector state polarization  $\epsilon_\nu$ . In full QCD the electromagnetic current of quarks is conserved, while in NRQCD the current  $\mathcal{J}_\nu^{NRQCD}$  has the nonzero anomalous dimension, so that in accordance with (1)–(5), we find

$$\langle 0 | \mathcal{J}_\nu^{NRQCD}(\mu) | \bar{Q}Q, \lambda \rangle = \mathcal{A}(\mu) \epsilon_\nu^\lambda f_{\bar{Q}Q}^{NRQCD} M_{\bar{Q}Q}, \quad (7)$$

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<sup>1</sup>We use ordinary notations for the invariants of  $SU(N_c)$  representations:  $C_F = \frac{N_c^2 - 1}{2N_c}$ ,  $C_A = N_c$ ,  $T_F = \frac{1}{2}$ ,  $n_f$  is a number of “active” light quark flavors.

where, in terms of nonrelativistic quarks, the leptonic constant for the heavy quarkonium is given by the well-known relation with the wave function at the origin

$$f_{\bar{Q}Q}^{NRQCD} = \sqrt{\frac{12}{M}} |\Psi_{\bar{Q}Q}(0)|, \quad (8)$$

and the value of wave function in the leading order is determined by the solution of Schrödinger equation with the static potential, and therefore, it is the renormalization invariant quantity in contrast to the factor of  $\mathcal{A}(\mu)$ . It is evident that

$$f_{\bar{Q}Q} = f_{\bar{Q}Q}^{NRQCD} \mathcal{A}(\mu_{\text{fact}}) \cdot \mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}}), \quad (9)$$

and the anomalous dimension of  $\mathcal{A}(\mu_{\text{fact}})$  should compensate the anomalous dimension of factor  $\mathcal{K}(\mu_{\text{hard}}; \mu_{\text{fact}})$ , so that in two loops we have got

$$\frac{d \ln \mathcal{A}(\mu)}{d \ln \mu} = -\gamma_{[2]} \left( \frac{\alpha_s^{\overline{\text{MS}}}(\mu)}{4\pi} \right)^2. \quad (10)$$

The physical meaning of  $\mathcal{A}(\mu)$  is clearly determined by the relations of (7) and (9): this factor gives the normalization of matrix element for the current of nonrelativistic quarks expressed in terms of wave function for the two-particle quark state (in the leading order of inverse heavy quark mass in NRQCD). Certainly, in this approach the current of nonrelativistic quarks is factorized from the quark-gluon sea, which is a necessary attribute of hadronic state, so that, in general, this physical state can be only approximately represented as the two-quark bound state. In the consideration of leptonic constants in the framework of NRQCD, this approximation requires to introduce the normalization factor  $\mathcal{A}(\mu)$  depending on the scale.

The renormalization group equation of (10) is simply integrated out, so that

$$\mathcal{A}(\mu) = \mathcal{A}(\mu_0) \left[ \frac{\beta_0 + \beta_1 \frac{\alpha_s^{\overline{\text{MS}}}(\mu)}{4\pi}}{\beta_0 + \beta_1 \frac{\alpha_s^{\overline{\text{MS}}}(\mu_0)}{4\pi}} \right]^{\frac{\gamma_{[2]}}{2\beta_1}}, \quad (11)$$

where  $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_F n_f$ , and  $\beta_1 = \frac{34}{3}C_A^2 - 4C_F T_F n_f - \frac{20}{3}C_A T_F n_f$ . A constant of integration could be defined so that at a scale  $\mu_0$  we would get  $\mathcal{A}(\mu_0) = 1$ . Thus, in the framework of NRQCD we have got the parametric dependence of leptonic constant estimates on the scale  $\mu_0$ , which has the following simple interpretation: the normalization of matrix element for the current of nonrelativistic quarks at  $\mu_0$  is completely given by the wave function of two-quark bound state. At other  $\mu \neq \mu_0$  we have to introduce the factor  $\mathcal{A}(\mu) \neq 1$ , so that the approximation of hadronic state by the two-quark one becomes inexact.

The relations above are general for the NRQCD description of leptonic constant for the heavy quarkonium in both the approach of sum rules and the calculations with the static potential, so that the only difference is the use of (8) in the potential approach. In the sum rules this relation is substituted by the equations for the constants  $f_{\bar{Q}Q}^{NRQCD}$ , which follow from the quark-hadron duality in the corresponding scheme of calculations. In this respect, we have to stress that the analysis of NRQCD sum rules for the bottomonium in the two-loop approximation in [5, 7, 16] took into account the factor of  $\mathcal{A}(\mu)$  in a different way. So, in the calculation of two-point correlator for the

currents of nonrelativistic quarks the regularization is necessary in the two-loop order. Of course, the anomalous dimension of two-point current correlator compensates the anomalous dimension of corresponding Wilson coefficient. In the sum rules with the one-loop gluon correction [4] this factor can be omitted, since its one-loop anomalous dimension is equal to zero, and, hence, the factor can be supposed equal to unit. As we have shown taking into account the two-loop corrections, the calculations without the normalization factor  $\mathcal{A}(\mu)$  for the matrix element of NRQCD current would be, in general, not correct.

### 3 Potential approach

The potential of static heavy quarks illuminates the most important features of QCD dynamics: the asymptotic freedom and confinement. In the leading order of perturbative QCD at short distances and with a linear confining term in the infrared region, the potential of static heavy quarks was considered in the Cornell model [17], incorporating the simple superposition of both asymptotic limits (the effective coulomb and string-like interactions). The observed heavy quarkonia posed in the intermediate distances, where both terms are important for the determination of mass spectra (see Fig. 1). So, the phenomenological approximations of potential (logarithmic one [18] and power law [19]), taking into account the regularities of such the spectra, were quite successful [20].

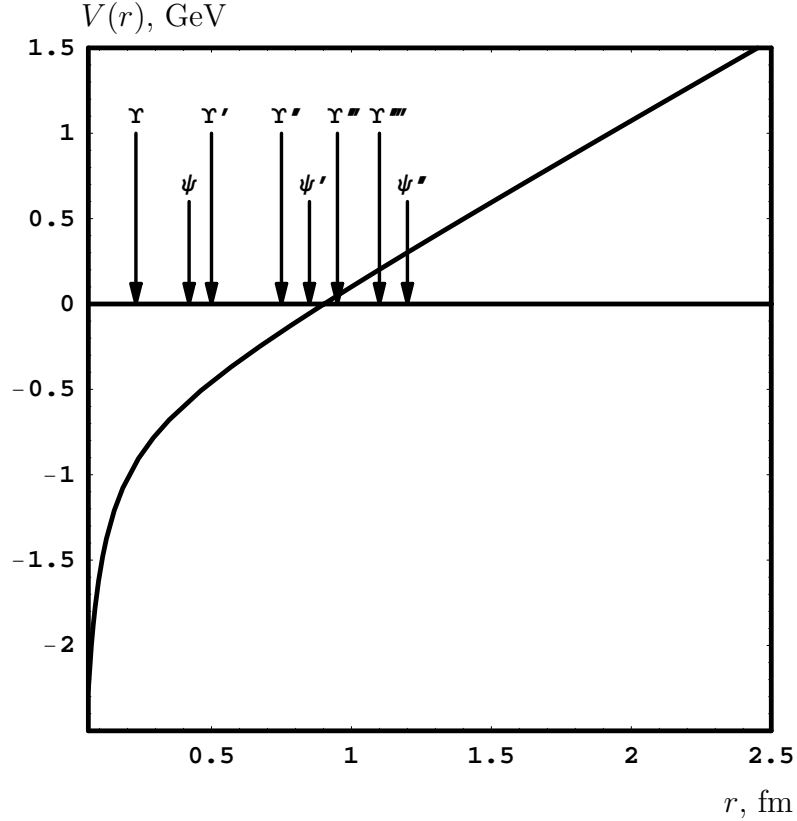


Figure 1: The Cornell model of static potential and sizes of observed heavy quarkonia with charmed quarks (the family  $\psi$ ) and bottom quarks (the family  $\Upsilon$ ).

The quantities more sensitive to the global properties of potential are the wave functions at

the origin as related to the leptonic constants and production rates. So, the potentials consistent with the asymptotic freedom to one and two loops as well as the linear confinement were proposed by Richardson [21], Buchmüller and Tye [12], respectively.

In QCD the static potential is defined in a manifestly gauge invariant way by means of the vacuum expectation value of a Wilson loop [22],

$$\begin{aligned} V(r) &= -\lim_{T \rightarrow \infty} \frac{1}{iT} \ln \langle \mathcal{W}_\Gamma \rangle, \\ \mathcal{W}_\Gamma &= \tilde{\text{tr}} \mathcal{P} \exp \left( ig \oint_\Gamma dx_\mu A^\mu \right). \end{aligned} \quad (12)$$

Here,  $\Gamma$  is taken as a rectangular loop with time extension  $T$  and spatial extension  $r$ . The gauge fields  $A_\mu$  are path-ordered along the loop, while the color trace is normalized according to  $\tilde{\text{tr}}(..) = \text{tr}(..)/\text{tr}\mathbb{1}$ . This definition corresponds to the calculation of effective action for the case of two external sources fixed at a distance  $r$  during an infinitely long time period  $T$ , so that the time-ordering coincides with the path-ordering. Moreover, the contribution into the effective action by the path parts, where the charges have been separated to the finite distance during a finite time, can be neglected in comparison with the infinitely growing term of  $V(r) \cdot T$ . Let us emphasize that the defined static potential is, by construction, the renormalization invariant quantity, since the action, by definition, does not depend on the normalization point.

Generally, one introduces the  $V$  scheme of QCD coupling constant by the definition of QCD potential of static quarks in momentum space as follows:

$$V(\mathbf{q}^2) = -C_F \frac{4\pi\alpha_v(\mathbf{q}^2)}{\mathbf{q}^2}, \quad (13)$$

so that for the such-way introduced value of  $\alpha_v$  one can derive some results at large virtualities in the perturbative QCD as well as at low transfer momenta in the approximation of linear term in the potential confining the quarks.

In this section, first, we discuss two regimes for the QCD forces between the static heavy quarks: the asymptotic freedom and confinement. Then we follow the method by Buchmüller and Tye and formulate how these regimes can be combined in a unified  $\beta$  function for  $\alpha_v$  obeyed both limits of small and large QCD couplings.

In the perturbative QCD the quantity  $\alpha_v$  can be matched with  $\alpha_{\overline{\text{MS}}}$

$$\alpha_v(\mathbf{q}^2) = \alpha_{\overline{\text{MS}}}(\mu^2) \sum_{n=0}^{\infty} \tilde{a}_n(\mu^2/\mathbf{q}^2) \left( \frac{\alpha_{\overline{\text{MS}}}(\mu^2)}{4\pi} \right)^n = \alpha_{\overline{\text{MS}}}(\mathbf{q}^2) \sum_{n=0}^{\infty} a_n \left( \frac{\alpha_{\overline{\text{MS}}}(\mathbf{q}^2)}{4\pi} \right)^n. \quad (14)$$

At present, in expansion (14) the coefficients of tree approximation  $a_0$ , one-loop contribution  $a_1$  and new two-loop results  $a_2$  (see [10, 11]) are known.

After the introduction of  $\mathbf{a} = \frac{\alpha}{4\pi}$ , the  $\beta$  function is actually defined by

$$\frac{d\mathbf{a}(\mu^2)}{d \ln \mu^2} = \beta(\mathbf{a}) = - \sum_{n=0}^{\infty} \beta_n \cdot \mathbf{a}^{n+2}(\mu^2), \quad (15)$$

so that  $\beta_{0,1}^V = \beta_{0,1}^{\overline{\text{MS}}}$  and  $\beta_2^V = \beta_2^{\overline{\text{MS}}} - a_1 \beta_1^{\overline{\text{MS}}} + (a_2 - a_1^2) \beta_0^{\overline{\text{MS}}}$ .

The Fourier transform results in the position-space potential [10], so that the perturbative potential, by construction, is independent of normalization point, i.e. it is the renormalization

group invariant. However, in the problem under consideration the truncation of perturbative expansion, wherein the coefficients do not decrease<sup>2</sup>, leads to a strong custodial dependence on the normalization point. So, putting the normalization point  $\mu$  in the region of charmed quark mass, we find that the two-loop potential with the three-loop running coupling constant  $\alpha_s^{\overline{\text{MS}}}$  has an unremovable additive shift depending on  $\mu$ . This shift has variation in wide limits. This fact illuminates the presence of infrared singularity in the coupling constant of QCD, so that the  $\mu$ -dependent shift in the potential energy has the form of a pole posed at  $\Lambda_{QCD}$  [9].

Thus, in order to avoid the ambiguity of static potential in QCD we have to deal with infrared stable quantities. The motivation by Buchmüller and Tye was to write down the  $\beta$  function of  $\alpha_v$  consistent with two known asymptotic regimes at short and long distances. They proposed the function, which results in the effective charge determined by two parameters, only: the perturbative parameter is the scale in the running of coupling constant at large virtualities and the nonperturbative parameter is the string tension. The necessary inputs are the coefficients of  $\beta$  function. The parameters of potential by Buchmüller and Tye were fixed by fitting the mass spectra of charmonium and bottomonium [3]. Particularly, in such the phenomenological approach the scale  $\Lambda_{\overline{\text{MS}}}^{n_f=4} \approx 510$  MeV was determined. It determines the asymptotic behaviour of coupling constant at large virtualities in QCD. This value is in a deep contradiction with the current data on the QCD coupling constant  $\alpha_s^{\overline{\text{MS}}}$  [3]. In addition, one can easily find that the three-loop coefficient  $\beta_2^v$  for the  $\beta$  function suggested by Buchmüller and Tye is not correct even by its sign and absolute value in comparison with the exact coefficient recently calculated in [10, 11].

Thus, the modification of Buchmüller–Tye (BT) potential of static quarks as dictated by the current status of perturbative calculations is of great interest.

The nonperturbative behaviour of QCD forces between the static heavy quarks at long distances  $r$  is usually represented by the linear potential (see discussion in ref. [23])

$$V^{\text{conf}}(r) = k \cdot r, \quad (16)$$

which corresponds to the square-law limit for the Wilson loop. The form of (16) corresponds to the limit, when at low virtualities  $\mathbf{q}^2 \rightarrow 0$  the coupling  $\alpha_v$  tends to

$$\alpha_v(\mathbf{q}^2) \rightarrow \frac{K}{\mathbf{q}^2},$$

so that

$$\frac{d\alpha_v(\mathbf{q}^2)}{\ln \mathbf{q}^2} \rightarrow -\alpha_v(\mathbf{q}^2), \quad (17)$$

which gives the confinement asymptotics for the  $\beta_v$  function.

Buchmüller and Tye proposed the procedure for the reconstruction of  $\beta$  function in the whole region of charge variation by the known limits of asymptotic freedom to a given order in  $\alpha_s$  and confinement regime. Generalizing their method, the  $\beta_{\text{PT}}$  function found in the framework of asymptotic perturbative theory (PT) to three loops, is transformed to the  $\beta$  function of effective charge as follows:

$$\frac{1}{\beta_{\text{PT}}(\mathbf{a})} = -\frac{1}{\beta_0 \mathbf{a}^2} + \frac{\beta_1 + \left(\beta_2^v - \frac{\beta_1^2}{\beta_0}\right) \mathbf{a}}{\beta_0^2 \mathbf{a}} \Rightarrow$$

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<sup>2</sup>Moreover, according to the investigations of renormalon, the coefficients in the series of perturbation theory for the potential increase in the factorial power, so that the series has a meaning of asymptotic one.

$$\frac{1}{\beta(\mathbf{a})} = -\frac{1}{\beta_0 \mathbf{a}^2 \left(1 - \exp\left[-\frac{1}{\beta_0 \mathbf{a}}\right]\right)} + \frac{\beta_1 + \left(\beta_2^V - \frac{\beta_1^2}{\beta_0}\right) \mathbf{a}}{\beta_0^2 \mathbf{a}} \exp\left[-\frac{l^2 \mathbf{a}^2}{2}\right], \quad (18)$$

where the exponential factor in the second term contributes to the next-to-next-to-leading order at  $\mathbf{a} \rightarrow 0$ . This function has the essential peculiarity at  $\mathbf{a} \rightarrow 0$ , so that the expansion is the asymptotic series in  $\mathbf{a}$ . At  $\mathbf{a} \rightarrow \infty$  the  $\beta$  function tends to the confinement limit represented in (17).

The construction of (18) is based on the idea to remove the pole from the coupling constant at a finite energy, but in contrast to the “analytic” approach developed in [24], the “smoothing” of peculiarity occurs in the logarithmic derivative of charge related with the  $\beta$  function, but in the expression for the charge itself. Indeed, in the one-loop perturbation theory we have got

$$\frac{d \ln \mathbf{a}}{d \ln \mu^2} = -\beta_0 \mathbf{a} = -\frac{1}{\ln \frac{\mu^2}{\Lambda^2}},$$

because of

$$\mathbf{a} = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}},$$

and the pole can be cancelled in the logarithmic derivative itself, so that<sup>3</sup>

$$\frac{d \ln \mathbf{a}}{d \ln \mu^2} \Rightarrow -\beta_0 \mathbf{a} \left(1 - \frac{\Lambda^2}{\mu^2}\right) \approx -\beta_0 \mathbf{a} \left(1 - \exp\left[-\frac{1}{\beta_0 \mathbf{a}}\right]\right).$$

As we see in the perturbative limit  $\mathbf{a} \rightarrow 0$  the deviation in the  $\beta$  function is exponentially small. The generalization of this idea to two loops was done by Buchmüller and Tye.

Remember that the one and two-loop static potentials matched with the linear term of confinement lead to the contradiction with the value of QCD coupling constant extracted at the scale of  $Z$  boson mass if we fit the mass spectra of heavy quarkonia in such potentials. The static potential in the three-loop approximation results in the consistent value of QCD coupling constant at large virtualities [9].

In the perturbative limit the usual solution for the running coupling constant

$$\mathbf{a}(\mu^2) = \frac{1}{\beta_0 \ln \frac{\mu^2}{\Lambda^2}} \left[ 1 - \frac{\beta_1}{\beta_0^2} \frac{1}{\ln \frac{\mu^2}{\Lambda^2}} \ln \ln \frac{\mu^2}{\Lambda^2} + \frac{\beta_1^2}{\beta_0^4} \frac{1}{\ln^2 \frac{\mu^2}{\Lambda^2}} \left( \ln^2 \ln \frac{\mu^2}{\Lambda^2} - \ln \ln \frac{\mu^2}{\Lambda^2} - 1 + \frac{\beta_2^V \beta_0}{\beta_1^2} \right) \right], \quad (19)$$

is valid. Using the asymptotic limit of (19), one can get the equation

$$\ln \frac{\mu^2}{\Lambda^2} = \frac{1}{\beta_0 \mathbf{a}(\mu^2)} + \frac{\beta_1}{\beta_0^2} \ln \beta_0 \mathbf{a}(\mu^2) + \int_0^{\mathbf{a}(\mu^2)} dx \left[ \frac{1}{\beta_0 x^2} - \frac{\beta_1}{\beta_0^2 x} + \frac{1}{\beta(x)} \right], \quad (20)$$

which can be easily integrated out, so that we get an implicit solution for the charge depending on the scale. The implicit equation can be inverted by the iteration procedure, so that well approximated solution has the form

$$\mathbf{a}(\mu^2) = \frac{1}{\beta_0 \ln \left(1 + \eta(\mu^2) \frac{\mu^2}{\Lambda^2}\right)}, \quad (21)$$

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<sup>3</sup>The presentation of power correction in the form of charge exponential was similarly used in [25].



where  $\eta(\mu^2)$  is expressed through the coefficients of perturbative  $\beta$  function and parameter  $l$  in (18), which is related to the slope of Regge trajectories  $\alpha'_P$  and the integration constant, the scale  $\Lambda$  [9].

The slope of Regge trajectories, determining the linear part of potential, is supposed equal to  $\alpha'_P = 1.04 \text{ GeV}^{-2}$ , so that in (16) we put the parameter  $k = \frac{1}{2\pi\alpha'_P}$ . We use also the measured value of QCD coupling constant [3] and pose

$$\alpha_s^{\overline{\text{MS}}}(m_Z^2) = 0.123,$$

as the basic input of the potential. The transformation into the configuration space was done numerically in [9], so that the potential is presented in the form of file in the notebook format of MATHEMATICA system.

The analysis of quark masses and mass spectra of heavy quarkonia results in the following values ascribed to the potential approach [9]:

$$m_c^V = 1.468 \text{ GeV}, \quad m_b^V = 4.873 \text{ GeV}. \quad (22)$$

Thus, the spectroscopic characteristics of systems composed of nonrelativistic heavy quarks are determined in the approach with the static potential described above.

## 4 Numerical estimates

In the potential approach the values of wave functions for the  $1S$  states in the systems of  $\bar{b}b$  and  $\bar{c}c$  are equal to

$$\sqrt{4\pi}\Psi_{\bar{b}b}(0) = 2.513 \text{ GeV}^{3/2}, \quad \sqrt{4\pi}\Psi_{\bar{c}c}(0) = 0.895 \text{ GeV}^{3/2}, \quad (23)$$

which are the renormalization group invariants by construction of Schrödinger equation with the static potential. Remember that this potential includes the contribution of linear part providing the the confinement of quarks. The same term makes the infrared stability of effective charge and all of the potential in NRQCD.

The analysis of leptonic constants of  $\Upsilon$  and  $\psi$  with account of Wilson coefficient  $\mathcal{K}$ , derived in the perturbative NRQCD, was done in [9]. For  $\Upsilon$  we found that the variation of hard scale in wide limits  $\mu_{\text{hard}} = (1 - 2)m_b$  led to the presence of a stable point  $\mu_{\text{fact}}$ , in which the result was weakly sensitive to the variation of hard scale. The stability was observed at  $\mu_{\text{fact}} \approx 2.2 - 2.7 \text{ GeV}$ , i.e. at the inter-quark distances characteristic for the size of  $1S$  level in the system of  $\bar{b}b$ .

For the leptonic constant of charmonium  $J/\psi$  the stability point under the variation of  $\mu_{\text{hard}}$  was located at the reasonable values of soft scale of factorization  $\mu_{\text{fact}} \approx 1.05 - 1.35 \text{ GeV}$ . The position of stable point was significantly dependent of changes in  $\mu_{\text{hard}}$  close to the mass of charmed quark.

However, in that consideration the stability of results under the variation of scale  $\mu_{\text{fact}}$  for the perturbative calculations concerning the Wilson coefficient in NRQCD was not observed.

In this paper we introduce the normalization factor  $\mathcal{A}$  into the analysis for the current of nonrelativistic quarks. As we see in Fig. 2, in the system of  $\bar{b}b$  the choice of initial point of normalization  $\mu_0 = 2.3 \text{ GeV}$  in the mentioned region of stability for the leptonic constant under the variation of matching scale, so that  $\mathcal{A}(2.3 \text{ GeV}) = 1$ , we get the independence of result for the lepton constant of  $\Upsilon$  in the interval  $\mu_{\text{fact}} = 1.8 - 3.8 \text{ GeV}$ .

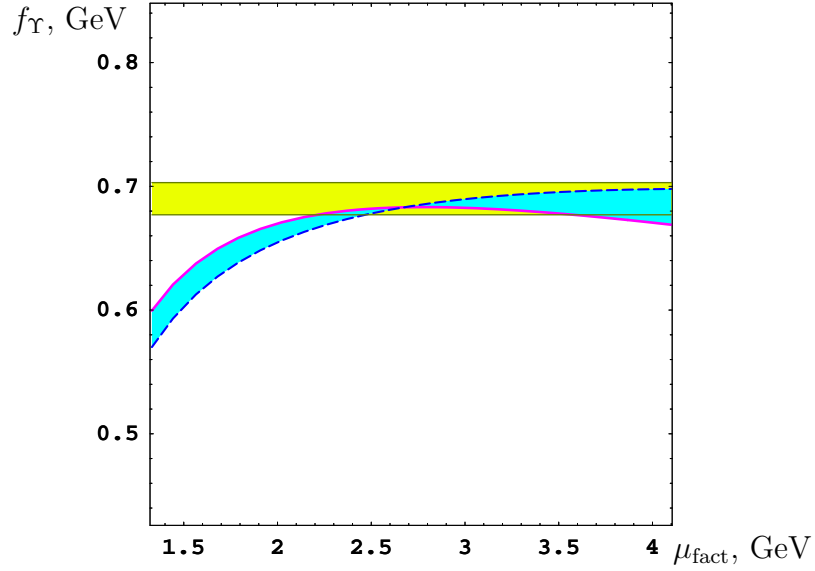


Figure 2: The leptonic constant of ground vector state in the system of bottomonium is presented versus the soft scale of normalization. The dashed curve corresponds to  $\mu_{\text{hard}} = 2m_b$ , while the solid curve does to  $\mu_{\text{hard}} = m_b$ . The initial condition for the evolution of normalization factor  $\mathcal{A}(\mu_{\text{fact}})$  in the matrix element of current in the nonrelativistic representation has been posed in the form  $\mathcal{A}(2.3 \text{ GeV}) = 1$ . The horizontal band is the experimental limits for the constant.

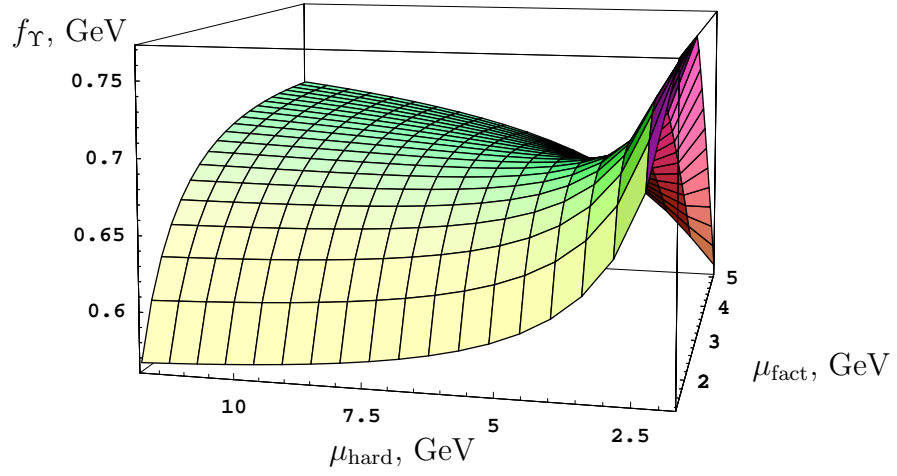


Figure 3: The leptonic constant of ground vector state in the system of bottomonium is presented versus the soft and hard scales,  $\mu_{\text{fact}}$  and  $\mu_{\text{hard}}$ . The initial condition for the evolution of normalization factor  $\mathcal{A}(\mu_{\text{fact}})$  in the matrix element of current in the nonrelativistic representation is  $\mathcal{A}(2.3 \text{ GeV}) = 1$ .

The analysis of stability is performed in Fig. 3. Thus, our estimate of leptonic constant for the ground vector state of bottomonium is given by

$$f_{\Upsilon} = 685 \pm 30 \text{ MeV},$$

which is in a good agreement with the experimental data.

The calculations of leptonic constant for the charmonium  $J/\psi$  are more sensitive to the choice of initial condition in the evolution of factor  $\mathcal{A}$ , because the stability point under the variation of hard scale for the matching is significantly fluctuates under the change of  $\mu_{\text{hard}} = (1 - 2) m_c$ . In Fig. 4 we show the dependence of  $f_{J/\psi}$  on the scale  $\mu_{\text{fact}}$  at  $\mathcal{A}(1.07 \text{ GeV}) = 1$ . This value of  $\mu_0 = 1.07 \text{ GeV}$  agrees with the inverse size of system  $\bar{c}c$ . We see in the figure that the region of result stability is given by  $\mu_{\text{fact}} = 1. - 1.6 \text{ GeV}$ .

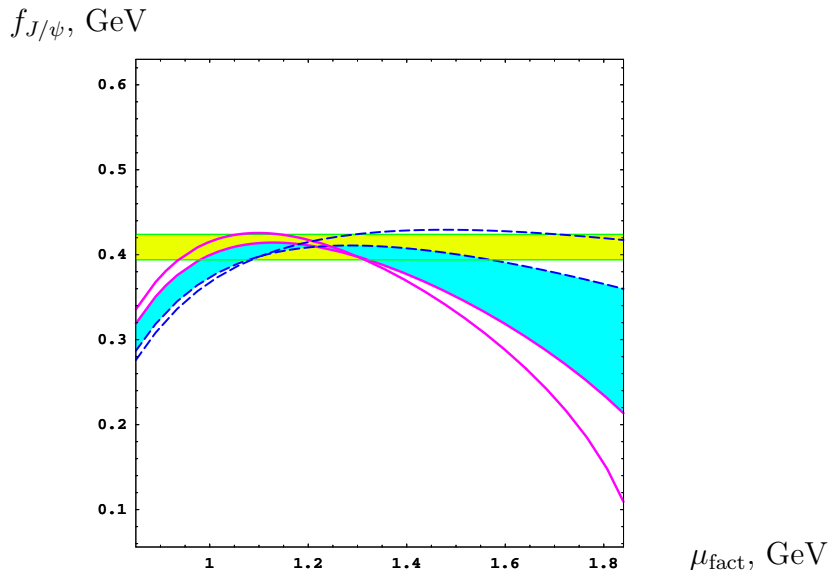


Figure 4: The leptonic constant of ground vector state in the system of charmonium is presented versus the soft scale of normalization. The shaded region restricted by curves corresponds to the change of hard scale from  $\mu_{\text{hard}} = 2.0 m_c$  (the dashed curve) to  $\mu_{\text{hard}} = 1.3 m_c$  (the solid curve). The initial condition for the evolution of normalization factor  $\mathcal{A}(\mu_{\text{fact}})$  in the matrix element of current in the nonrelativistic representation has been posed in the form  $\mathcal{A}(1.07 \text{ GeV}) = 1$ . The horizontal band is the experimental limits for the constant. The additional curves are presented for  $\mu_{\text{hard}} = 2.8 m_c$  (the dashed line) and  $\mu_{\text{hard}} = 1.15 m_c$  (the solid line).

The dependence of estimates on the scales are presented in Fig. 5. Then we get

$$f_{J/\psi} = 400 \pm 45 \text{ MeV},$$

which can be compared with the experimental value  $f_{J/\psi}^{\text{exp}} = 409 \pm 15 \text{ MeV}$ .

Thus, the analysis of stability for the leptonic constants of heavy quarkonia in the framework of potential approach shows that there is the region of stability under the change of both factorization scale in the perturbative NRQCD and point marking the matching of effective theory of nonrelativistic quarks with the full QCD. The basic source of uncertainty in the estimates is the variation of initial normalization point for the evolution factor determining the matrix element for the current of nonrelativistic quarks, so that the values of leptonic constants in the extremum (see

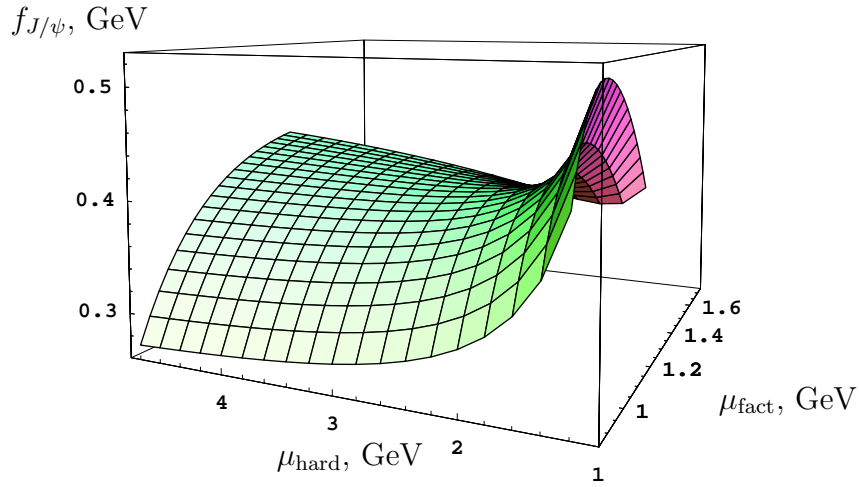


Figure 5: The leptonic constant of ground vector state in the system of charmonium is presented versus the soft and hard scales,  $\mu_{\text{fact}}$  and  $\mu_{\text{hard}}$ . The initial condition for the evolution of normalization factor  $\mathcal{A}(\mu_{\text{fact}})$  in the matrix element of current in the nonrelativistic representation is given by  $\mathcal{A}(1.07 \text{ GeV}) = 1$ .

Figs. 3 and 5) change in the limits, which are included into the methodic error of estimates as shown above.

Finally, in the potential approach we can calculate the ratios of leptonic constants for the excited  $nS$ -wave levels in  $\bar{b}b$  and  $\bar{c}c$ , which were presented in [9] in comparison with the experimental data. These theoretical calculations are in a good agreement with the measured values.

## 5 Conclusion

In the present work we have done the general consideration of problem on the calculation of leptonic constants for the heavy quarkonia in NRQCD and introduced the factor for the normalization of matrix element for the current of nonrelativistic quarks. This factor has got the anomalous dimension of current in the effective theory, so that the explicit invariance under the action of renormalization group takes place for the estimated values of leptonic constants. In the framework of potential approach we have estimated the constants for  $f_\Upsilon$  and  $f_{J/\psi}$ , so that we have found reasonable regions of stability under the variation of both the scale for the matching of NRQCD with full QCD and the soft scale ascribed to the operators of nonrelativistic quarks in NRQCD. The obtained results are in a good agreement with the experimental data.

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